Tensor Products of Classifiable C^*-algebras

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Abstract

Let $\mathcal{A}_1$ be the class of all unital separable simple C^*-algebras A such that $A\otimes U$ has tracial rank at most one for all UHF-algebras of infinite type. It has been shown that amenable $\mathcal{Z}$-stable C^*-algebras in $\mathcal{A}_1$ which satisfy the Universal Coefficient Theorem can be classified up to isomorphism by the Elliott invariant. We show that $A\in\mathcal{A}_1$ if and only if $A\otimes B$ has tracial rank at most one for one of unital simple infinite dimensional AF-algebra B. In fact, we show that $A\in\mathcal{A}_1$ if and only if $A\otimes B\in\mathcal{A}_1$ for some unital simple AH-algebra B. Other results regarding the tensor products of C^*-algebras in $\mathcal{A}_1$ are also obtained.

1 Introduction

The Elliott program of classification of amenable C^*-algebras is to classify separable amenable C^*-algebras up to isomorphisms by its K-theoretic data known as the Elliott invariants. It is a very successful program. Two important classes of unital separable simple C^*-algebras, the class of amenable separable purely infinite simple C^*-algebras satisfying the Universal Coefficient Theorem (UCT) and unital simple AH-algebras with no dimension growth are classified by their Elliott invariants (see [9] and [5] and [6] among many literatures). There are a number of other significant progress in the Elliott program. Related to this note, it has been shown that unital separable amenable simple C^*-algebras with tracial rank at most one which satisfy the UCT are classifiable by the Elliott invariants and they are isomorphic to unital simple AH-algebras with no dimension growth. More recently it was shown in [17] that unital separable amenable simple $\mathcal{Z}$-stable C^*-algebras which satisfy the UCT and are rationally tracial rank at most one are also classifiable by the Elliott invariants (see also [19] and [26]). This class is significantly larger than the class of all unital simple AH-algebras with no dimension growth. A unital separable simple C^*-algebra A is said to be rationally tracial rank at most one if $A\otimes U$ has tracial rank at most one for every UHF-algebra U of infinite type. Denote by $\mathcal{A}_1$ the class of all unital separable simple C^*-algebra which are rationally tracial rank at most one. A special unital separable simple C^*-algebra in $\mathcal{A}_1$ which does not have finite tracial rank is the Jiang-Su algebra $\mathcal{Z}$. The range of the Elliott invariant for rationally tracial rank at most one has been characterized and computed ([18]). This class of C^*-algebras includes C^*-algebras whose ordered K_0-groups may not have the Riesz interpolation property. The verification that a particular unital simple C^*-algebra is in the class $\mathcal{A}_1$ was slightly eased when it was proved in [18] that, $A\in\mathcal{A}_1$ if and only if $A\otimes U$ has tracial rank at most one for some UHF-algebra U of infinite type (instead for all UHF-algebras of infinite type). Suppose A is a unital separable simple C^*-algebra such that $A\otimes B$ has tracial rank at most one for some unital infinite dimensional simple AF-algebra B. Does it follow that $A\in\mathcal{A}_1$? We will answer this question affirmatively in this short note. In fact, we will show that if $A\otimes B$ has tracial rank at most one for some unital infinite dimensional separable simple C^*-algebra B with tracial rank at most one then $A\in\mathcal{A}_1$. This may provide a better way to determine which C^*-algebras are in $\mathcal{A}_1$.

For the classification purpose, we also consider $\mathcal{C}_1$ the class of all unital separable simple amenable C^*-algebras which are rationally tracial rank at most one and which satisfy the UCT.

We will show that if A and B are both in $\mathcal{C}_1$, then $A \otimes B \in \mathcal{C}_1$. Now suppose that $A \in \mathcal{C}_1$ and B has tracial rank at most one. Then, from the above, $A \otimes B$ is also in $\mathcal{C}_1$. One may also ask whether $A \otimes B$ has tracial rank at most one? We will give an affirmative answer to this question.

2 Preliminaries

Definition 2.1. let A be a C^*-algebra. Let $\mathcal{F}$ and $\mathcal{G}$ be two subsets. Let $\epsilon > 0$. We say that $\mathcal{F} \subset_\epsilon \mathcal{G}$ if for each $x \in \mathcal{F}$, there exists $y \in \mathcal{G}$, such that $\|x - y\| < \epsilon$.

If $a, b \in A_+$ are two elements in a C^*-algebra A, we write $a \lesssim b$ if there exists $x \in A$ such that $xx^* = a$ and $x^*x \in \overline{bAb}$.

In C^*-algebra A, let $\mathcal{F} \subset A$ be a finite subset and let $p \in A$ be a projection. We use $p\mathcal{F}p$ to denote $\{pxp\colon x \in \mathcal{F}\}$. Let B be a subalgebra of A and let $\epsilon > 0$. We write $\mathcal{F} \subset_\epsilon B$ if $\mathrm{dist}(x, B) < \epsilon$ for all $x \in \mathcal{F}$.

Definition 2.2. Denote by $\mathcal{I}_1$ the class of all finite direct sums of C^*-algebras of the form $M_n(C([0,1]))$ (for different integers $n \in \mathbb{N}$).

Recall that a unital simple C^*-algebra A has tracial rank at most one, if the following holds: For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $a \in A_+ \setminus \{0\}$, there exists a projection $p \in A$ and there exists a C^*-subalgebra $B \subset A$ with $B \in \mathcal{I}_1$ and $1_B = p$ such that

$$\|px - xp\| < \epsilon \text{ for all } x \in \mathcal{F}, \tag{e 2.1}$$

$$p\mathcal{F}p \subset_\epsilon B \text{ and} \tag{e 2.2}$$

$$1 - p \lesssim a. \tag{e 2.3}$$

Note that, in definition 2.7 of [13], $\mathcal{I}_1$ in the above is replaced by the class of all finite direct sums of C^*-algebras of the form $M_n(C(X))$, where X is one of finite CW complexes with dimension 1. According to Theorem 7.1 of [13], they are equivalent. If, in the above, $\mathcal{I}_1$ is replaced by $\mathcal{I}_0$, the class of finite dimensional C^*-algebras, then A has tracial rank zero. If A has tracial rank at most one, we write $TR(A) \leq 1$. If A has tracial rank zero, we write $TR(A) = 0$.

Notations: Let A be a unital C^*-algebra. Denote by $M_\infty(A)$ the set of all finite rank matrices over A. Denote by $T(A)$ the tracial state space of A. If $p \in M_\infty(A)$, then $p \in M_n(A)$ for some integer $n \geq 1$. We write $\tau(p)$ for $(\tau \otimes Tr)(p)$, where Tr is the standard trace on M_n.

Denote by $\mathcal{N}$ the class of all unital separable amenable C^*-algebras which satisfy the Universal Coefficient Theorem.

Denote by Q the UHF-algebra with $(K_0(Q), K_0(Q)_+, [1_Q]) = (\mathbb{Q}, \mathbb{Q}_+, 1)$.

We use $\mathcal{A}_0$ to denote the class of all unital separable simple C^*-algebras A for which $TR(A \otimes M_\mathfrak{p}) = 0$ for all supernatural numbers $\mathfrak{p}$ of infinite type

Use $\mathcal{A}_1$ to denote the class of all unital separable simple C^*-algebras A for which $TR(A \otimes M_\mathfrak{p}) \leq 1$ for all supernatural numbers $\mathfrak{p}$ of infinite type.

Use $\mathcal{C}_0$ to denote the class of all unital separable simple amenable C^*-algebras A in $\mathcal{N}$ for which $TR(A \otimes M_\mathfrak{p}) = 0$ for all supernatural numbers $\mathfrak{p}$ of infinite type.

Use $\mathcal{C}_1$ to denote the class of all unital separable simple amenable C^*-algebras A in $\mathcal{N}$ for which $TR(A \otimes M_\mathfrak{p}) \leq 1$ for all supernatural numbers $\mathfrak{p}$ of infinite type.

Definition 2.3. Let A be a unital C^*-algebra with $T(A) \neq \emptyset$. We say that A has the property of strict comparison for projections, if $\tau(p) < \tau(q)$ for all $\tau \in T(A)$ implies that $p \lesssim q$ for all projections in $M_\infty(A)$.

3 Criterions for C^*-algebras to be rationally tracial rank at most one

Theorem 3.1. *Let A be a unital simple separable C^*-algebra, and let C be a unital infinite dimensional simple AF-algebra. Suppose that $A\otimes C$ has tracial rank at most one. Then $A\in\mathcal{A}_1$.*

Proof. Put $B=A\otimes Q$. Let $\epsilon>0$, a nonzero element $a\in B_+\setminus\{0\}$ and a finite subset $\mathcal{F}\subset B$ be given. We may assume that $\|a\|=1$ and $\epsilon<1/4$.

We will write $A\otimes Q=\lim_{k\to\infty}(A\otimes M_{k!},j_k)$, where $j_k: A\otimes M_{k!}\to A\otimes M_{(k+1)!}$ by $j_k(a)=a\otimes 1_{M_{(k+1)}}$ for all $a\in A\otimes M_{k!}$, $k=1,2,....$ Without loss of generality, we may assume that $\mathcal{F}\subset A\otimes M_{k!}$ for some $k\ge 1$.

Without loss of generality, we may assume that there exists a positive element $a'\in A\otimes M_{k!}$ such that $\|a-a'\|<\epsilon$. Let

$$f_\epsilon(t)=\begin{cases}1 & t\ge 2\epsilon\\ (1/\epsilon)t-1 & \epsilon<t<2\epsilon\\ 0 & t\le\epsilon\end{cases}.$$

According to Proposition 2.2 and Lemma 2.3 (b) of [21], $f_\epsilon(a')\lesssim a$. Put $a_0=f_\epsilon(a')$. As $\|a\|=1$, $\epsilon<1/4$, it is clear that $a_0\in(A\otimes M_{k!})_+\setminus\{0\}$.

Write C as $\lim_{m\to\infty}(C_m,\imath'_m)$, where C_m is a finite dimensional C^*-algebra and where $\imath'_m: C_m\to C_{m+1}$, is a unital embedding. Since C is an infinite dimensional unital simple AF-algebra, we may write that

$$C_m=M_{n_1}\oplus M_{n_2}\oplus\cdots\oplus M_{n_{s(m)}}, \tag{e 3.4}$$

where $n_j\ge k!$, $j=1,2,...,s(m)$ for all large m. Fix one of such m. Thus one obtains a projection $q\in C_m$ so that $M_{k!}$ is a unital C^*-subalgebra of qC_mq. Put $e=1\otimes q$ and let $\varphi'_1: M_{k!}\to qC_mq$ be a unital embedding. Define $\varphi_1: A\otimes M_{k!}\to A\otimes qC_mq$ by $\varphi_1(a\otimes b)=a\otimes\varphi'_1(b)$ for all $a\in A$ and $b\in M_{k!}$.

It follows from Theorem 3.6 of [15] that $e(A\otimes C)e$ has tracial rank no more that one. Therefore there exists a projection $p\in e(A\otimes C)e$ and a C^*-subalgebra $I_0\in\mathcal{I}_1$ (interval C^*-algebras) of $e(A\otimes C)e$ with $1_{I_0}=p$,

$$\|px-xp\| < \epsilon/2 \text{ for all } x\in\varphi_1(\mathcal{F}), \tag{e 3.5}$$
$$\mathrm{dist}(pxp,I_0) < \epsilon/2 \text{ for all } x\in\varphi_1(\mathcal{F}) \text{ and} \tag{e 3.6}$$
$$1-p \lesssim \varphi_1(a_0). \tag{e 3.7}$$

Let $K_0=\max_{x\in\mathcal{F}}\{\|x\|\}$ and let $K=\max(K_0,16)$. Choose a finite set $\mathcal{G}_0$ in I_0, such that $p\mathcal{F}p\subset_{\epsilon/16}\mathcal{G}_0$. Let $\mathcal{G}_1$ be a finite generator set of I_0 and let $\mathcal{G}=\mathcal{G}_0\cup\mathcal{G}_1\cup\{1_{I_0}\}$. Since I_0 is weakly semi-projective, according to Lemma 15.2.1 of [20], for n large enough, there exists a homomorphism $h\colon I_0\to A\otimes(qC_nq)$ such that $\|h(y)-y\|<\epsilon/K$ for all $y\in\mathcal{G}$. In particular, $h(p)$ is a projection in $A\otimes(qC_nq)$ such that $\|p-h(p)\|<\epsilon/8$. As $\epsilon<1/4$, we know that p and $h(p)$ are untarily equivalent.

It can be checked that

$$\|h(p)x-xh(p)\| < 5\epsilon/8 \text{ for all } x\in\varphi_1(\mathcal{F}), \tag{e 3.8}$$
$$\mathrm{dist}(h(p)xh(p),h(I_0)) < 11\epsilon/16 \text{ for all } x\in\varphi_1(\mathcal{F}) \text{ and} \tag{e 3.9}$$
$$1-h(p) \lesssim \varphi_1(a_0). \tag{e 3.10}$$

To simplify notation, we may assume that $I_0\subset A\otimes qC_nq$, where $p\in A\otimes qC_nq$ for some large n satisfying $n\ge m$. Write $qC_nq=M_{m_1}\oplus M_{m_2}\oplus\cdots M_{m_r}$. Note that $k!|m_j$ for $j=1,2,...,r$, as φ_1' is unital. Put $N = \sum_{j=1}^r m_j$. Therefore there is a unital embedding $\varphi_2' : qC_nq \to M_{N!}$. Consider $j_k : M_{k!} \to M_{N!}$ and $\varphi_2' \circ \varphi_1' : M_{k!} \to M_{N!}$. Since they both are unital, there is a unitary $u \in M_{N!}$ such that

$$\mathrm{Ad}u \circ \varphi_2' \circ \varphi_1' = j_k.$$

Define $\varphi_2 : A \otimes qC_nq \to A \otimes M_{N!}$ by

$$\varphi_2(a \otimes b) = a \otimes (\mathrm{Ad}\, u \circ \varphi_2'(b))$$

for all $a \in A$ and $b \in qC_nq$.

Then

$$(\varphi_2 \circ \varphi_1)(c) = c \text{ for all } c \in A \otimes M_{k!}. \tag{e 3.11}$$

Put $p_1 = \varphi_2(p) \in A \otimes M_{N!} \subset A \otimes Q$ and $D = \varphi_2(I_0) \subset A \otimes M_{N!} \subset A \otimes Q$ with $1_D = p_1$. Note also $D \in \mathcal{I}_1$ (interval algebras). Moreover, by (e 3.5) and (e 3.6), we have

$$\|p_1x - xp_1\| = \|\varphi_2(p\varphi_1(x) - \varphi_1(x)p)\| = \|p\varphi_1(x) - \varphi_1(x)p\| < \epsilon/2 \text{ for all } x \in \mathcal{F}; \tag{e 3.12}$$
$$\mathrm{dist}(p_1xp_1, D) \le \mathrm{dist}(p\varphi_1(x)p, I_0) < \epsilon/2 \text{ for all } x \in \mathcal{F}. \tag{e 3.13}$$

Moreover, by (e 3.7),

$$1 - p_1 = \varphi_2(1-p) \lesssim \varphi_2(\varphi_1(a)) = a. \tag{e 3.14}$$

This implies that $TR(A \otimes Q) \le 1$, which shows that $A \in \mathcal{A}_1$.

□

Lemma 3.2. *Let A be a unital separable simple C^*-algebra and let C be a unital simple AH-algebra with $\mathrm{Tor}(K_0(C)) = \{0\}$ and with no dimension growth. Suppose that $A \otimes C$ has tracial rank no more than one. Then $A \in \mathcal{A}_1$.*

Proof. Note that $\mathrm{Tor}(K_0(C)) = \{0\}$, by Lemma 8.1 of [7] , $K_0(C)$ is an unperforated Riesz group. It follows from the Effros-Handelman-Shen theorem (Theorem 2.2 of [2]) that there exists a unital separable simple AF-algebra B with

$$(K_0(B), K_0(B)_+, [1_B]) = (K_0(C), K_0(C)_+, [1_C]). \tag{e 3.15}$$

We will show that $TR(A \otimes B) \le 1$. For that, let $\epsilon > 0$, $\mathcal{F} \subset A \otimes B$ be a finite subset and let $a \in (A \otimes B)_+ \setminus \{0\}$. Without loss of generality, we may assume that $1/2 > \epsilon$, $\mathcal{F}$ is a subset of the unit ball and $\|a\| = 1$.

We may assume that there are $a_{f,1}, a_{f,2}, ...a_{f,n(f)} \in A$ and $b_{f,1}, b_{f,2}, ..., b_{f,n(f)} \in B$ such that

$$\|f - \sum_{i=1}^{n(f)} a_{f,i} \otimes b_{f,i}\| < \epsilon/16 \text{ for all } f \in \mathcal{F}. \tag{e 3.16}$$

We may also assume that there exist $x_1, x_2, ..., x_{n(a)} \in A$ and $y_1, y_2, ..., y_{n(a)} \in B$ such that

$$\|a - \sum_{i=1}^{n(a)} x_i \otimes y_i\| < \epsilon/16. \tag{e 3.17}$$

Let

$$K_1 = n(a) + \max\{n(f) : f \in \mathcal{F}\}, \tag{e 3.18}$$
$$K_2 = \max\{\|x_i\| + \|y_i\| : 1 \le i \le n(a)\} \text{ and} \tag{e 3.19}$$
$$K_3 = \max\{\|a_{f,i}\| + \|b_{f,i}\| : 1 \le i \le n(f) \text{ and } f \in \mathcal{F}\}. \tag{e 3.20}$$

Put $a_1 = f_\epsilon(a)$ with f_ϵ as defined in the proof of Theorem 3.1.

As B is an AF-algebra and C has stable rank one, it is known that there exists a unital homomorphism $\varphi_1' : B \to C$ such that $(\varphi_1')_*$ gives the identification (e 3.15). Define $\varphi_1 : A\otimes B \to A\otimes C$ by $\varphi_1 = \mathrm{id}_A \otimes \varphi_1'$. Now since $TR(A\otimes C) \le 1$, there exists a projection $q \in A\otimes C$ and a C^*-subalgebra $D \in \mathcal{I}_1$ such that $1_D = p$ and

$$\begin{aligned} \|px - xp\| &< \epsilon/16 \text{ for all } x \in \varphi_1(\mathcal{F}), && \text{(e 3.21)}\\ \mathrm{dist}(pxp, D) &< \epsilon/16 \text{ for all } x \in \varphi_1(\mathcal{F}) \text{ and} && \text{(e 3.22)}\\ 1 - p &\lesssim \varphi_1(a_1). && \text{(e 3.23)} \end{aligned}$$

Let $\kappa \in KL(C, B)$ such that $\kappa|_{K_1(C)} = 0$ and $\kappa|_{K_0(C)} = (\varphi_1')_{*0}^{-1}$. It follows from [14] that there exists a unital embedding $\varphi_2' : C \to B$ such that

$$[\varphi_2'] = \kappa. \tag{e 3.24}$$

Let

$$\mathcal{G} = \{y_i : 1 \le i \le n(a)\} \cup \{b_{f,i} : 1 \le i \le n(f) \text{ and } f \in \mathcal{F}\}. \tag{e 3.25}$$

Put

$$\delta = \frac{\epsilon}{16 K_1 K_2 K_3}. \tag{e 3.26}$$

Note that

$$[\varphi_2' \circ \varphi_1'] = [\mathrm{id}_B] \text{ in } KL(B, B). \tag{e 3.27}$$

According to Lemma 4.2 of [3], there exists a unitary $u \in B$ such that

$$\|(\mathrm{Ad}\, u \circ \varphi_2' \circ \varphi_1')(y) - y\| < \delta \text{ for all } y \in \mathcal{G}. \tag{e 3.28}$$

Define $\varphi_2 : A\otimes C \to A\otimes B$ by $\varphi_2 = \mathrm{id}_A \otimes (\mathrm{Ad}\, u \circ \varphi_2')$. Put $p_1 = \varphi_2(p)$ and $D_1 = \varphi_2(D)$.

Then, one estimates, by (e 3.28) and (e 3.16), that

$$\|\varphi_2 \circ \varphi_1(f) - f\| < \epsilon/16 + \epsilon/16 + K_1 K_3 \delta < 3\epsilon/16 \text{ for all } f \in \mathcal{F}. \tag{e 3.29}$$

Similarly,

$$\|\varphi_2 \circ \varphi_1(a) - a\| < \epsilon/16 + \epsilon/16 + K_1 K_2 \delta < 3\epsilon/16 \tag{e 3.30}$$

Thus, we have, by applying (e 3.21) and (e 3.29), that

$$\begin{aligned} \|p_1 x - x p_1\| &\le \|p_1 x - \varphi_2(p)\varphi_2\circ\varphi_1(x)\| && \text{(e 3.31)}\\ &\quad + \|\varphi_2(p)\varphi_2\circ\varphi_1(x) - \varphi_2\circ\varphi_1(x)\varphi_2(p)\| && \text{(e 3.32)}\\ &\quad + \|\varphi_2\circ\varphi_1(x)\varphi_2(p) - x p_1\| && \text{(e 3.33)}\\ &< 3\epsilon/16 + \|p\varphi_1(x) - \varphi_1(x)p\| + 3\epsilon/16 < 7\epsilon/16 && \text{(e 3.34)} \end{aligned}$$

for all $x \in \mathcal{F}$. Similarly,

$$\mathrm{dist}(p_1 x p_1, D_1) < 7\epsilon/16 \text{ for all } x \in \mathcal{F}. \tag{e 3.35}$$

Also, by (e 3.23),

$$1 - p_1 \lesssim \varphi_2(\varphi_1(a_1)). \tag{e 3.36}$$

In other words,

$$1 - p_1 \lesssim f_\epsilon(\varphi_2 \circ \varphi_1(a)). \tag{e 3.37}$$

From (e 3.30) and Proposition 2.2 and Lemma 2.3 (b) of [21], we have

$$f_\epsilon(\varphi_2 \circ \varphi_1(a)) \lesssim a. \tag{e 3.38}$$

It follows that

$$1 - p_1 \lesssim a. \tag{e 3.39}$$

This proves that $A \otimes B$ has tracial rank no more than one.

□

Theorem 3.3. *Let A be a unital separable simple C^*-algebra. Suppose that $TR(A \otimes C) \leq 1$ for some unital amenable separable simple C^*-algebra C such that $TR(C) \leq 1$ and C satisfies the UCT. Then $A \in \mathcal{A}_1$.*

Proof. Suppose that $TR(A \otimes C) \leq 1$. We may assume that C has infinite dimension. Otherwise, as C is simple, $C \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. With $TR(M_n(A)) \leq 1$, according to Theorem 3.6 in [15], $TR(A) \leq 1$. Therefore $A \in \mathcal{A}_1$.

Now assume that C is infinitely dimensional. As $TR(A \otimes C) \leq 1$, we have $TR((A \otimes C) \otimes Q) \leq 1$. Note that $(A \otimes C) \otimes Q \cong A \otimes (C \otimes Q)$. As $TR(C) \leq 1$, it follows that $TR(C \otimes Q) \leq 1$. Since C is amenable and satisfies UCT, $C \otimes Q$ is also a unital separable amenable simple C^*-algebra which satisfies the UCT. It follows from Theorem 10.4 of [15] that $C \otimes Q$ is a unital simple AH-algebra with no dimension growth. One computes that $K_0(C \otimes Q)$ is torsion free. It follows from Lemma 3.2 that $A \in \mathcal{A}_1$. □

Corollary 3.4. *Let A be a unital separable simple C^*-algebra. Suppose that $TR(A \otimes C) = 0$ for some unital amenable separable simple C^*-algebra with $TR(C) \leq 1$ which satisfies the UCT. Then $A \in \mathcal{A}_0$.*

Proof. The proof is similar to that of Theorem 3.3. □

Corollary 3.5. *Let A be a unital separable simple C^*-algebra. Suppose that $TR(A \otimes C) \leq 1$ for some unital simple AH-algebra C. Then $A \in \mathcal{A}_1$.*

Proof. Note that by Theorem 10.4 of [15] , $C \otimes Q$ is a unital simple AH-algebra with no dimension growth. Since $TR(A \otimes C) \leq 1$, $TR(A \otimes C \otimes Q) \leq 1$. □

Proposition 3.6. *Let A be a unital separable simple C^*-algebra. Then the following are quivalent:*

1) $A \in \mathcal{A}_1$.
2) $A \otimes U \in \mathcal{A}_1$ for some UHF-algebra U of infinite type.
3) $A \otimes U \in \mathcal{A}_1$ for any UHF-algebra U of infinite type.

Proof. "3) ⇒ 2)" is obvious.

"2) ⇒ 1)": Suppose that $A \otimes U \in \mathcal{A}_1$ for some UHF-algebra U of infinite type. Then $TR(A \otimes U \otimes U) \leq 1$. But $(A \otimes U) \otimes U \cong A \otimes U$. We conclude that $TR(A \otimes U) \leq 1$. This implies 1) holds.

"1) ⇒ 3)": Since $A \in \mathcal{A}_1$, $TR(A \otimes U) \leq 1$ for any UHF-algebra U of infinite type. In particular, $A \otimes U \in \mathcal{A}_1$.

□

4 Tensor Products

Proposition 4.1. *Let A and B be two unital amenable separable simple C^*-algebras in $\mathcal{C}_1$. Then $A \otimes B \in \mathcal{C}_1$.*

Proof. Let $A, B \in \mathcal{C}_1$. Then

$$(A \otimes B) \otimes Q \cong (A \otimes B) \otimes (Q \otimes Q) \cong (A \otimes Q) \otimes (B \otimes Q).$$

Since both A and B are in $\mathcal{C}_1$, $A \otimes Q$ and $B \otimes Q$ have tracial rank no more than one and satisfy the UCT. Therefore each of them is isomorphic to some unital simple AH-algebars with no dimension growth. It is then easy to see that $(A \otimes Q) \otimes (B \otimes Q)$ can be written as a unital simple AH-algebra with no dimension growth, which implies that $TR(A \otimes B \otimes Q) \leq 1$.

□

Theorem 4.2. *Let A be a unital separable simple C^*-algebra. Suppose that there exists a unital separable simple amenable C^*-algebra $B \in \mathcal{C}_1$ such that $A \otimes B \in \mathcal{A}_1$, then $A \in \mathcal{A}_1$.*

Proof. Since $A \otimes B \in \mathcal{A}_1$, $TR(A \otimes B \otimes Q) \leq 1$. As $B \in \mathcal{C}_1$, we have that $B \otimes Q$ satisfies UCT and $TR(B \otimes Q) \leq 1$. By Lemma 10.9 and Theorem 10.10 of [12], $B \otimes Q$ is a unital simple AH-algebra with no dimension growth. Note that $\mathrm{Tor}(K_0(B \otimes Q)) = 0$. It follows from Lemma 3.2 that $A \in \mathcal{A}_1$.

□

We now consider the converse of Theorem 3.3 in the following sense. Let $A \in \mathcal{A}_1$. Is it true that $TR(A \otimes C) \leq 1$ if C is a unital separable infinite dimensional simple C^*-algebra with $TR(C) \leq 1$?

Definition 4.3. Let A be a unital separable simple C^*-algebra. We say A has the property of tracially approximate divisibility, if the following holds: For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$, any $a \in A_+ \setminus \{0\}$, any integer $N \geq 1$, there exists a projection $p \in A$, a finite dimensional C^*-subalgebra $B \subset A$ with $1_B = p$ and

$$B = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k}$$

such that $n_j \geq N$ for $j = 1, 2, ..., k$, and

$$\|px - xp\| < \epsilon \text{ for all } x \in \mathcal{F} \quad \text{(e 4.40)}$$
$$\|b(pxp) - (pxp)b\| < \epsilon \text{ for all } x \in \mathcal{F}, b \in B \text{ with } \|b\| \leq 1 \text{ and} \quad \text{(e 4.41)}$$
$$1 - p \lesssim a. \quad \text{(e 4.42)}$$

It is proved in Theorem 5.4 of [12] that every unital infinite dimensional separable simple C^*-algebra A with $TR(A) \leq 1$ is tracially approximately divisible.

Lemma 4.4. *Let A be a unital infinite dimensional separable C^*-algebra and let B be a unital separable simple C^*-algebra which is tracially approximately divisible. Then, for any non-zero projection $p \in A \otimes B$ and any integer $n \geq 1$, there are $n+1$ mutually orthogonal non-zero projections $p_1, p_2, ..., p_n$ and p_{n+1} such that p is equivalent to $\sum_{j=1}^{n+1} p_j$ and $[p_1] = [p_2] = \cdots = [p_n]$.*

Proof. Let $C = A \otimes B$. Then pCp is a unital infinite dimensional simple C^*-algebra. Therefore pCp contains a positive element $0 \leq f_0 \leq 1$ with infinite spectrum. From here, using the fact that pCp is simple, one obtains two non-zero mutually orthogonal elements $f_1, f_2 \lesssim f_0$ such that $2[f_1] \le [p]$ in the Cuntz semigroup. Fix $n \ge 1$. There are $K \in \mathbb{N}$, $a_1, a_2, ..., a_K \in A$ and $b_1, b_2, ..., b_K \in B$ such that

$$\|p - \sum_{i=1}^{K} a_i \otimes b_i\| < 1/64(n+1)^2. \qquad \text{(e 4.43)}$$

Let $M = K^2 \max\{(\|a_i\| + 1)(\|b_i\| + 1) : 1 \le i \le K\}$. Since B is tracially approximately divisible, there is a projection $e \in B$ and $D \subset B$ with $1_D = e$,

$$D = M_{n_1} \oplus M_{n_2} \oplus \cdots M_{n_m}$$

with $n_j \ge 4n+1$, $j = 1, 2, ..., m$, such that

$$\begin{aligned} \|eb_j - b_j e\| &< 1/64(M+1)(n+1)^2, j = 1, 2, ..., K, & \text{(e 4.44)}\\ \|deb_j e - eb_j ed\| &< 1/64(M+1)(n+1)^2 & \text{(e 4.45)}\\ & \quad \text{for all } d \in D \text{ with } \|d\| \le 1, \ \ j = 1, 2, ..., K & \text{(e 4.46)}\\ \text{and } 1_B - e &\lesssim f_1. & \text{(e 4.47)} \end{aligned}$$

It follows that

$$\begin{aligned} \|ep - pe\| &< 1/64(n+1)^2, & \text{(e 4.48)}\\ \|depe - eped\| &< 1/64(n+1)^2 & \text{(e 4.49)}\\ & \quad \text{for all } d \in D \text{ with } \|d\| \le 1, \ \ j = 1, 2, ..., L & \text{(e 4.50)}\\ \text{and } 1_{A\otimes B} - 1_A \otimes e &\lesssim f_1. & \text{(e 4.51)} \end{aligned}$$

One notes that $epe \ne 0$, by (e 4.51).

It is easy to produce a projection $e_0 \in D$ and $n-1$ unitaries $u_1, u_2, ..., u_{n-1} \in D$ such that $e_0, u_j^* e_0 u_j$ (for $j = 1, 2, ..., n-1$) are mutually orthogonal projections in D. By (e 4.50), one obtains a projection p_1, unitaries $v_1, v_2, ..., v_{n-1}$ such that

$$p_1, v_1^* p_1 v_1, v_2^* p_1 v_2, ..., v_{n-1}^* p_1 v_{n-1} \qquad \text{(e 4.52)}$$

are mutually orthogonal projections such that

$$\begin{aligned} & \|p_1 - e_0 p e_0\| < 1/16(n+1)^2, & \text{(e 4.53)}\\ & \|v_j^* p_1 v_j - u_j^* e_0 p e_0 u_j\| < 1/16(n+1), \ , \ \ j = 1, 2, ..., n-1 \ \text{ and} & \text{(e 4.54)}\\ & \|e_1 p e_1 - (p_1 \oplus \sum_{j=1}^{n-1} v_j^* p_1 v_j)\| < 1/16, & \text{(e 4.55)} \end{aligned}$$

where $e_1 = e_0 \oplus \sum_{j=1}^{n-1} u_j^* e_0 u_j \in D$. Define $p_{j+1} = v_j^* p_1 v_j$, $j = 1, 2, ..., n-1$. There is a projection $p_{n+1} \in (1 - e_1)(A \otimes B)(1 - e_1)$ such that

$$\|p_{n+1} - (1 - e_1)p(1 - e_1)\| < 1/16(M+1)(n+1)^2. \qquad \text{(e 4.56)}$$

One verifies that p is equivalent to $\sum_{k=1}^{n+1} p_k$. The lemma then follows.

□

Lemma 4.5. *Let A be a unital separable infinite dimensional simple C^*-algebra with $T(A) \ne \emptyset$ and B be a unital separable simple C^*-algebra which is tracially approximately divisible and also has at least one tracial state. Suppose that $A \otimes B$ has the strictly comparison for projections. Then, for any non-zero projections $p, q \in A \otimes B$ and any integer $n \ge 1$, there are $n+1$ mutually orthogonal non-zero projections $p_1, p_2, ..., p_n, p_{n+1}$ such that $p = p_1 + \ldots + p_n + p_{n+1}$, p_j is equivalent to p_1 for $j = 1, 2, ..., n$, $p_{n+1} \lesssim p_1$ and $p_{n+1} \lesssim q$.*

Proof. The proof is similar to that of 4.4. Put $C = A \otimes B$. Choose an integer $m \geq 1$ such that

$$\inf\{\tau(q) : \tau \in T(C)\} > \frac{1}{m+n}. \tag{e 4.57}$$

By Lemma 4.4, there is a non-zero projection $p_0 \leq p$ such that $\tau(p_0) < \frac{1}{4(m+n)}$ for all $\tau \in T(C)$.

Let $d = \inf\{\tau(p_0) : \tau \in T(A)\} > 0$.

Since B is tracially approximately divisible, as in the proof of Lemma 4.4, there exists a projection $e \in B$ and $D \subset B$ with $1_D = e$,

$$D = M_{r_1} \oplus M_{r_2} \oplus \cdots \oplus M_{r_k}$$

with

$$\frac{n}{r_j} < d/2, \;\; j = 1, 2, ..., k \tag{e 4.58}$$

such that

$$\|ep - pe\| < \frac{1}{64(n+1)^2} \tag{e 4.59}$$

$$\|d(epe) - (epe)d\| < \frac{1}{64(n+1)^2} \text{ for all } d \in D \text{ with } \|d\| \leq 1 \tag{e 4.60}$$

and $$1 - e \lesssim p_0. \tag{e 4.61}$$

By (e 4.58), there is a projection e_0 and $n-1$ uniatries $u_1, u_2, ..., u_{n-1} \in D$ such that $e_0, u_1^* e_0 u_1, u_2^* e_0 u_2, ..., u_{n-1}^* e_0 u_{n-1}$ are mutually orthogonal projections in D such that

$$\tau(e - (e_0 + \sum_{j=1}^{n-1} u_j^* e_0 u_j)) < d \text{ for all } \tau \in T(A). \tag{e 4.62}$$

It follows that

$$(e - (e_0 + \sum_{j=1}^{n-1} u_j^* e_0 u_j)) \lesssim p_0. \tag{e 4.63}$$

We then obtain a projection $p_1 \in C$ and unitaries $v_1, v_2, ..., v_{n-1}$ such that p_1, $v_1^* p_1 v_1$, ..., $v_{n-1}^* p_1 v_{n-1}$ are mutually orthogonal and

$$\|p_1 - e_0 p e_0\| < 1/16(n+1), \tag{e 4.64}$$

$$\|v_j^* p_1 v_j - u_j^* e_0 p e_0 u_j\| < 1/16(n+1) \text{ and} \tag{e 4.65}$$

$$\|e_1 p e_1 - (p_1 + \sum_{j=1}^{n-1} v_j^* p_1 v_j)\| < 1/16, \tag{e 4.66}$$

where $e_1 = e_0 + \sum_{j=1}^{n-1} u_j^* e_0 u_j \in D$. There is also a projection $p_{n+1} \in (1-e_1)C(1-e_1)$ such that

$$\|p_{n+1} - (1-e_1)p(1-e_1)\| < 1/64(n+1)^2. \tag{e 4.67}$$

Put $p_{j+1} = v_j^* p_1 v_j$, $j = 1, 2, ..., n-1$. Thus, we have

$$[p] = [\sum_{j=1}^{n+1} p_j]. \tag{e 4.68}$$

We see that $p_1, p_2, ..., p_n$ are equivalent. Since C has strictly comparison, by (e 4.57), (e 4.58) and (e 4.62),

$$p_{n+1} \lesssim p_1 \text{ and } p_{n+1} \lesssim q.$$

□

Lemma 4.6. *Let $A \in \mathcal{C}_1$. Suppose that B is a unital separable amenable simple C^*-algebra with $TR(B) \le 1$ and satisfies UCT. Then $K_0(A \otimes B)$ has the Reisz interpolation property.*

Proof. Since $A \in \mathcal{C}_1$ and $TR(B) \le 1$, by Theorem 4.1, $A \otimes B \in \mathcal{C}_1$. It follows from [18] that $K_0(A \otimes B)$ is rationally Reisz. In other words, if we have $x_1, x_2, y_1, y_2 \in K_0(A \otimes B)$ such that $x_i \le y_j$, $i, j = 1, 2$, then there exists $z \in K_0(A \otimes B)$ and there are integers $m, n \in \mathbb{N}$ such that

$$mx_i \le nz \text{ and } nz \le my_j, \ \ i, j = 1, 2. \tag{e 4.69}$$

Denote by $S_u(K_0(A \otimes B))$ the state space of $K_0(A \otimes B)$. If $mx_1 = nz = my_1$, we claim that $x_1 = y_1$. Otherwise $y_1 = x_1 + w$ for some $w \in \mathrm{Tor}(K_0(A \otimes B))$. But $x_1 \le y_1$. It would imply that $w \ge 0$. By 4.1 $A \otimes B \in \mathcal{C}_1$. It follows that $K_0(A \otimes B)$ is weakly unperforated. if $w \ne 0$, $s(w) > 0$ for all states $s \in S_u(K_0(A \otimes B))$. But this is impossible since $mw = 0$. Now if $x_1 = y_1$, set $z_1 = x_1$. Then

$$x_i \le y_1 = z_1 = x_1 \le y_j, \ \ i, j = 1, 2.$$

Let us consider the case that $mx_i \ne nz$, $i = 1, 2$. It follows that

$$s(x_i) < (n/m)s(z) \le s(y_j) \text{ for all } s \in S_u(K_0(A \otimes B)), \ \ i, j = 1, 2. \tag{e 4.70}$$

We may assume that $x_i \in K_0(A \otimes B)_+$ for $i = 1, 2$. It follows that $z \in K_0(A \otimes B)_+ \setminus \{0\}$. Note that $S_u(K_0(A \otimes B))$ is compact. There exists $1 > d > 0$ such that

$$s(x_i) < (n/m)s(z) - d < s(y_j) \text{ for all } s \in S_u(K_0(A \otimes B)), \ \ i, j = 1, 2. \tag{e 4.71}$$

By replacing z by kz for some $k \in \mathbb{N}$, if necessarily, we may assume that $0 < n/m < 1$. Then, by Lemma 4.5, there is $w \in K_0(A \otimes B)_+$ such that $nz = mw + w_0$ and

$$s(w_0) < d \text{ for all } s \in S_u(K_0(A \otimes B)). \tag{e 4.72}$$

Let $z_1 = mw$. Note that $n/m < 1$, we then have

$$s(x_i) < s(z_1) < s(y_j) \text{ for all } x \in S_u(K_0(A \otimes B)). \tag{e 4.73}$$

Note that, by Corollary 8.4 of [16], B is $\mathcal{Z}$-stable. It follows that $A \otimes B$ is $\mathcal{Z}$-stable. According to Corollary 4.10 of [22], we have that

$$x_i \le z_1 \le y_j \ \ i, j = 1, 2. \tag{e 4.74}$$

This shows that $K_0(A \otimes B)$ has the Riesz interpolation property.

□

Theorem 4.7. *Let $A \in \mathcal{C}_1$. Then, for any unital infinite dimensional simple AH-algebra B with slow dimension growth, $A \otimes B$ is a unital simple AH-algebra with no dimension growth.*

Proof. Since $A \in \mathcal{C}_1$, it follows from 4.1 that $A \otimes B \in \mathcal{C}_1$. By Lemma 4.6, $K_0(A \otimes B)$ has the Riesz interpolation property. Since B is an infinite dimensional simple AH-algebra, $K_0(A \otimes B) \ne \mathbb{Z}$. Moreover the canonical map $r\colon T(A \otimes B) \to S_u(K_0(A \otimes B))$ maps the extremal points to extremal points. It follows from [24] that there is a unital simple AH-algebra C with no dimension growth such that the Elliott invariant is exactly the same as that of $A \otimes B$. According to Theorem 10.4 of [15], we have that $A \otimes B \cong C$.

□

We end the note by the following summarization:

Theorem 4.8. *Let $A \in \mathcal{N}$ be a unital separable simple amenable C^*-algebra that satisfies the UCT. Then the following are equivalent.*

(1) $A \in \mathcal{C}_1$;

(2) $TR(A \otimes Q) \leq 1$;

(3) $A \otimes Q \in \mathcal{A}_1$;

(4) $TR(A \otimes B) \leq 1$ for some unital infinite dimensional simple AF-algebra B;

(5) $TR(A \otimes B) \leq 1$ for all unital simple infinite dimensional AF-algebras B;

(6) $A \otimes B \in \mathcal{A}_1$ for some unital simple infinite dimensional AF-algebra B;

(7) $A \otimes B \in \mathcal{A}_1$ for all unital simple infinite dimensional AF-algebras B;

(8) $TR(A \otimes B) \leq 1$ for some unital infinite dimensional simple AH-algebra B with no dimension growth;

(9) $TR(A \otimes B) \leq 1$ for all unital simple infinite dimensional AH-algebras B with no dimension growth;

(10) $A \otimes B \in \mathcal{A}_1$ for some unital simple infinite dimensional AH-algebra B with no dimension growth;

(11) $A \otimes B \in \mathcal{A}_1$ for all unital simple infinite dimensional AH-algebra B with no dimension growth;

(12) $A \otimes B \in \mathcal{A}_1$ for all unital simple infinite dimensional C^*-algebra B in $\mathcal{C}_1$;

(13) $A \otimes B \in \mathcal{A}_1$ for some unital simple infinite dimensional C^*-algebra $B \in \mathcal{C}_1$.

Proof. Note that "(1) $\Rightarrow$ (2)", "(2) $\Rightarrow$ (3)", "(5) $\Rightarrow$ (4)", "(4) $\Rightarrow$ (6)", "(7) $\Rightarrow$ (6)", "(9) $\Rightarrow$ (8)", "(9) $\Rightarrow$ (10)", "(11) $\Rightarrow$ (10)", "(11) $\Rightarrow$ (7)", "(12) $\Rightarrow$ (11)", "(12) $\Rightarrow$ (7)" and "(12) $\Rightarrow$ (13)" are straightforward from the statement.

That "(1) $\Rightarrow$ (5)" and "(1) $\Rightarrow$ (9)" follow from 4.7. To see that "(1) $\Rightarrow$ (12)," let $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_1$. Then $TR(B \otimes Q) \leq 1$. So $B \otimes Q$ is a unital simple infinite dimensional AH-algebra with no dimension growth. Since "(1) $\Rightarrow$ (9)", this implies that $TR(A \otimes (B \otimes Q)) \leq 1$. It follows that $A \otimes B \in \mathcal{A}_1$.

For "(13) $\Rightarrow$ (1)", one has $TR(A \otimes B \otimes Q) \leq 1$. It follows that $TR(A \otimes (B \otimes Q)) \leq 1$. Since $TR(B \otimes Q) \leq 1$, again, $B \otimes Q$ is a unital simple infinite dimensional AH-algebra with no dimension growth. It follows from 3.3 that $A \in \mathcal{A}_1$. As $A \in \mathcal{N}$, it is in $\mathcal{C}_1$.

That "(3) $\Rightarrow$ (1)" follows from 3.6 and "(4) $\Rightarrow$ (1) follows from 3.1.

For "(6) $\Rightarrow$ (4)", one considers $A \otimes B \otimes Q$ and notes $B \otimes Q$ is a unital simple infinite dimensional AF-algebra.

That "(8) $\Rightarrow$ (4)" follows from 3.3.

The rest of implications follow similarly as established previously.

□

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